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RANKING THE FACETS OF THE OCTAHEDRON *

Egon BALAS

Management Sciences Research Group, Graduate School of Industrial Administration, Carnegie-Mellon University, Pittsburgh, Pa. 15213, USA

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Abstract. This paper describes a procedure for ranking the facets of the n-dimensional (regular) octahedron in the order in which they are intersected by a halfline. The problem discussed here arises in the context of integer programming via convex analysis (or intersection cuts).

§ 1. Problem statement

This paper is concerned with finding an efficient procedure for ranking the facets of the n-dimensional octahedron

$$K^* = \{x \in R^n \mid |x| \leq \tfrac{1}{2}n\},$$

where $|x| = \sum_{i=1}^n |x_i|$, in the order in which the affine hull of each facet is intersected by a given halfline

$$(1) \quad \xi = \{x \mid x = \bar{x} + a\lambda, \lambda \geq 0\}$$

originating at a point $\bar{x} \in \operatorname{int}(K^*)$, $\bar{x} = (\bar{x}_i)$, such that $-\frac{1}{2} \leq \bar{x}_i \leq \frac{1}{2}$, $i = 1, ..., n$, and having an arbitrary direction $a = (a_i)$. This problem arises in the context of the intersection cut (or convexity cut) approach to integer programming (see, for instance, [1]-[4]), where n halflines of the above type, originating at the linear programming optimum $\bar{x}$, are intersected with the boundary of some suitably chosen convex set in order to generate a valid cutting plane. The idea of intersecting the halflines ξ with (extended) facets of K^* other than the first facet met by

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each halfline, was first put forward by Fred Glover [4].

Since the octahedron K^* is the outer polar [2] of the unit cube

$$K = \{x \in R^n \mid -\tfrac{1}{2}e \leq x \leq \tfrac{1}{2}e\},$$

where $e \in R^n$, $e = (1, ..., 1)$, i.e. there is a one-to-one mapping that sends each vertex x of K into a facet $\varphi(x)$ of K^* such that $x \in \varphi(x)$, it should be noted that, when $\bar{x} = 0$, the above problem is equivalent to the trivial problem of ranking the vertices of K in the order in which they are intersected by the hyperplane $ax = z$ when z is decreased from some initial value $\hat{z} > \frac{1}{2}|a|$. When $\bar{x} \neq 0$, however, the equivalent problem in terms of K seems to be more difficult than the original problem formulated above in terms of K^*.

Each (extended) facet $f(K^*)$ of K^* is defined by an equation of the form

$$(2) \quad \delta x = \tfrac{1}{2}n,$$

where $\delta = (\delta_i)$, the normal to $f(K^*)$, is an n-vector defined by

$$(3) \quad \delta_i = \begin{cases} 1 & \text{if } i \in N^+, \\ -1 & \text{if } i \in N^-, \end{cases}$$

for some partition (N^+, N^-) of the index set $N = \{1, ..., n\}$. Two facets of K^* are called *adjacent* if their normals differ by exactly one component.

Given a halfline ξ of the form (1), the intersection $\xi \cap f(K^*) = \bar{x} - a\lambda$ of ξ with an arbitrary (extended) facet $f(K^*)$ of K^*, whose equation is (2), is then defined (if it exists) by

$$(4) \quad \lambda = \frac{\varphi}{\alpha} = \frac{\bar{x}\delta - \frac{1}{2}n}{a\delta}$$

for some $\delta = (\delta_i)$ of the form (3).

Note that $\varphi < 0$ for any δ of the form (3), since $-\frac{1}{2} \leq \bar{x}_i \leq \frac{1}{2}$, $i \in N$, with $-\frac{1}{2} < \bar{x}_i < \frac{1}{2}$ for at least one $i \in N$ (since $\bar{x} \in \operatorname{int}(K^*)$); hence $\delta\bar{x} < \frac{1}{2}n$. Thus $\lambda > 0$ implies $\alpha < 0$.

Given two (extended) facets f_1 and f_2 of K^*, such that $\xi \cap f_1 = \bar{x} - a\lambda^1$ and $\xi \cap f_2 = \bar{x} - a\lambda^2$, we say that f_1 is intersected *before* f_2

(f_2 is intersected *after* f_1) if $\lambda^1 < \lambda^2$. If f_1 and f_2 are two adjacent facets, the associated values λ^1 and λ^2 will also be called adjacent.

§2. Ranking the facets of K^*

Denote by $\dot\lambda_i$, $i \in N$, the n values adjacent to $\dot\lambda$, i.e.,

$$\dot\lambda_i = \frac{\dot\varphi - 2\bar{x}_i\delta_i}{\dot\alpha - 2a_i\delta_i}, \qquad i \in N .$$

We now define 3 subsets of N associated with $\dot\lambda$, namely

$$\begin{aligned} &N^1(\dot\lambda) = \{i \in N \mid \dot\lambda \le \dot\lambda_i < \infty\}, \\ (5) \qquad &N^2(\dot\lambda) = \{i \in N \mid 0 < \dot\lambda_i < \dot\lambda\}, \\ &N^3(\dot\lambda) = \{i \in N \mid \dot\lambda_i < 0 \text{ or } \dot\alpha = 2a_i\delta_i\}. \end{aligned}$$

It is easy to see that $\bigcup_{h=1}^3 N^h(\dot\lambda) = N$, $\bigcap_{h=1}^3 N^h(\dot\lambda) = \phi$.

Lemma 1. *Let N^* be an arbitrary subset of N. Further, let $N^* \cap N^1(\dot\lambda) \ne \phi$, and $\dot\lambda_k = \min\{\dot\lambda_i \mid i \in N^* \cap N^1(\dot\lambda)\}$. Then*

$$N^* \cap N^2(\dot\lambda_k) \subseteq N^* \cap [N^2(\dot\lambda) \cup \{k\}] .$$

Proof. Define

$$\begin{aligned} &\dot\varphi - 2\bar{x}_i\delta_i = \dot\varphi_i, \qquad \dot\alpha - 2a_i\delta_i = \dot\alpha_i, \quad i \in N, \\ (6) \qquad &\dot\varphi_k - 2\bar{x}_i\delta_i = \dot\varphi_{ki}, \quad \dot\alpha_k - 2a_i\delta_i = \dot\alpha_{ki}, \quad i \in N - \{k\}. \end{aligned}$$

From $k \in N^1(\dot\lambda)$, we have

$$\frac{\dot\varphi_k}{\dot\alpha_k} \ge \frac{\dot\varphi}{\dot\alpha} = \frac{\dot\varphi_k + 2\bar{x}_k\delta_k}{\dot\alpha_k + 2a_k\delta_k} .$$

Since $\dot\alpha_k < 0$, $\dot\alpha < 0$, multiplying through by $\dot\alpha_k(\dot\alpha_k + 2a_k\delta_k) > 0$ yields

(7) $2\bar{x}_k\delta_k\dot{\alpha}_k \le 2a_k\delta_k\dot{\varphi}_k$.

On the other hand, if $i \in N^2(\dot{\lambda}_k)$, then

$$\frac{\dot{\varphi}_k}{\dot{\alpha}_k} > \frac{\dot{\varphi}_{ki}}{\dot{\alpha}_{ki}} = \frac{\dot{\varphi}_{ik}}{\dot{\alpha}_{ik}} = \frac{\varphi_i - 2\bar{x}_k\delta_k}{\dot{\alpha}_i - 2a_k\delta_k} > 0,$$

or, multiplying through with $\dot{\alpha}_k(\dot{\alpha}_i - 2a_k\delta_k) > 0$,

(8) $\varphi_i\dot{\alpha}_k - 2\bar{x}_k\delta_k\dot{\alpha}_k < \dot{\varphi}_k\dot{\alpha}_i - 2a_k\delta_k\dot{\varphi}_k$.

Then, adding (7) to (8), we obtain

(9) $\dot{\varphi}_i\dot{\alpha}_k < \dot{\varphi}_k\dot{\alpha}_i$.

Since $\dot{\varphi}_i < 0$, $\dot{\varphi}_k < 0$ and $\dot{\alpha}_k < 0$, (9) implies $\dot{\alpha}_i < 0$ and hence

(10) $i \notin N^* \cap N^1(\dot{\lambda})$.

Further, dividing (9) by $\dot{\alpha}_k\dot{\alpha}_i > 0$, yields

$$\frac{\dot{\varphi}_i}{\dot{\alpha}_i} < \frac{\dot{\varphi}_k}{\dot{\alpha}_k} ,$$

which implies, in view of the definition of k,

(11) $i \notin N^* \cap N^1(\dot{\lambda}) - \{k\}$.

But then, from (10) and (11) we have $i \in N^* \cap [N^2(\dot{\lambda}) \cup \{k\}]$.

Theorem 1. *Let f_0 be the first (extended) facet of K^* intersected by ξ, with equation $\delta^0 x = \frac{1}{2}n$. Further, let f_p, with equation $\delta^p x = \frac{1}{2}n$, be an extended facet of K^* intersected by ξ after f_0, and such that δ^p differs from δ^0 in exactly k components, $1 < k < n$, i.e., $|N_0^*| = k$, where $N_0^* = \{i \in N \mid \delta_i^p \neq \delta_i^0\}$. Then there exists a sequence of $k+1$ (extended) facets $f_0, f_1, \ldots, f_k$ of K^*, with associated values $\lambda^0, \lambda^1, \ldots, \lambda^k$, defined by $\bar{x} - a\lambda^h = \xi \cap f_h$, $h = 0, 1, \ldots, k$, such that*

(i) $\lambda^0 \leq \lambda^1 \leq \ldots \leq \lambda^k$,

(ii) $f_k = f_p$,

(iii) *f_h is adjacent to f_{h-1}, $h = 1, \ldots, k$.*

Proof. Let $\delta^r_x = \frac{1}{2}n$ be the equation of f_r, $r = 1, \ldots, k$, and let

$$N^*_r = \{i \in N \mid \delta^p_i \neq \delta^r_i\} .$$

For all integers r, $0 \leq r \leq k-1$, define $N^h(\lambda^r)$, $h = 1, 2, 3$, as in (5), and let

$$\lambda^{r+1} = \lambda^r_{h(r)} = \min\{\lambda^r_i \mid i \in N^*_r \cap N^1(\lambda^r)\} .$$

We claim that λ^r is well-defined for $r = 0, 1, \ldots, k-1$, and will show this by induction on N^*_0. Then the theorem follows.

Since $\lambda^0 = \varphi_0/\alpha_0$ defines the intersection of ξ with the first (extended) facet of K^*, $N^2(\lambda^0) = \emptyset$ and $N^1(\lambda^0) \neq \emptyset$. We first show that $N^*_0 \cap N^1(\lambda^0) \neq \emptyset$. If this were false, then $N^*_0 \subseteq N^3(\lambda^0)$, i.e.

$$\alpha_i = \alpha_0 - 2a_i\delta^0_i \geq 0, \qquad \forall i \in N^*_0 ,$$

which, in view of $\alpha_0 < 0$ implies

$$\alpha_p = \alpha_0 - 2\sum_{i \in N^*_0} a_i\delta^0_i \geq 0 ,$$

hence $\lambda^p < 0$, or $\lambda^p = \infty$, contradicting the assumption that f_p is intersected by ξ. Hence $N^*_0 \cap N^1(\lambda^0) \neq \emptyset$, and thus $\lambda^1 = \lambda^0_{h(0)}$ is well-defined. Furthermore, from the definition of λ^0 and λ^1 we have $N^2(\lambda^0) = \emptyset$, $N^2(\lambda^1) = \{h(0)\}$, and since $N^*_1 = N^*_0 - \{h(0)\}$, it follows that $N^*_1 \cap N^2(\lambda^1) = \emptyset$, or

$$N^*_1 \subseteq N^1(\lambda^1) \cup N^3(\lambda^1) .$$

Suppose now that the two properties shown to hold for λ^1 also hold for λ^s, $s = 2, \ldots, r < k$, i.e., each λ^s is well-defined and

$$N_s^* \subseteq N^1(\lambda^s) \cup N^3(\lambda^s), \qquad s = 1, 2, \ldots, r.$$

Then λ^{r+1} is well-defined if and only if $N_r^* \cap N^1(\lambda^r) \neq \emptyset$. Suppose $N_r^* \cap N^1(\lambda^r) = \emptyset$, then $N_r^* \subseteq N^3(\lambda^r)$, i.e.

$$\alpha_r - 2\bar{a}_i\delta_i^r \geq 0, \qquad \forall i \in N_r^*.$$

But then, in view of $\alpha_r < 0$, we have

$$\alpha_p = \alpha_r - 2\sum_{i\in N_r^*} a_i\delta_i^r \geq 0,$$

which again contradicts the fact that f_p is intersected by ξ, i.e. that $\lambda^p = \varepsilon_p/\alpha_p > 0$. Hence λ^{r+1} is well-defined. Further, since $N_r^* \cap N^2(\lambda^r) = \emptyset$ by assumption, from Lemma 1

$$N_r^* \cap N^2(\lambda^{r+1}) \subseteq N_r^* \cap [N^2(\lambda^r) \cup \{h(r)\}] \subseteq \{h(r)\}$$

and thus from $N_{r+1}^* = N_r^* - \{h(r)\}$ we have

$$N_{r+1}^* \subseteq N^1(\lambda^r) \cup N^3(\lambda^r).$$

We have thus shown that $\lambda^0 \leq \lambda^1 \leq \ldots \leq \lambda^k$, where the associated sequence of vectors $\delta^0, \delta^1, \ldots, \delta^k$ is such that each vector δ^h of the sequence, $h = 1, \ldots, k$, differs from its predecessor by exactly one component. Hence each (extended) facet f_h, $h = 1, \ldots, k$, is adjacent to its predecessor in the associated sequence of facets $f_0, f_1, \ldots, f_k$, and $f_k = f_p$.

Corollary 1.1. *The equation $\delta x = \frac{1}{2}n$ defines the first (extended) facet of K^* intersected by ξ if and only if $N^2(\lambda) = \emptyset$, where λ and $N^2(\lambda)$ are defined by (4) and (5), respectively.*

Proof. The only if part is obvious. Let f be the (extended) facet of K^* defined by the equation $\delta x = \frac{1}{2}n$. Then $N^2(\lambda) = \emptyset$ if no (extended) facet adjacent to f is intersected by ξ before f. But from Theorem 1, if there is an (extended) facet of K^* intersected by ξ before f, then there is one intersected before f that is adjacent to f.

We now describe an algorithm for generating the values λ defining the intersection of ξ with all the extended facets of K^* met by ξ, in the order in which they are intersected. The procedure iteratively generates values λ defining the intersection of ξ with some extended facet of K^*, and orders them into a nondecreasing sequence, such that, at the kth iteration, the first k terms of the sequence define the intersection of ξ with the first k (extended) facets.

Whenever a value λ^s is generated, it is stored together with the associated vector $\delta^s \in R^n$ defining the facet intersected at $\bar{x} - a\lambda^s$, the relation between λ^s and δ^s being

$$(12) \qquad \lambda^s = \frac{\bar{x}\delta^s - \frac{1}{2}n}{a\delta^s}.$$

At each iteration k, we distinguish between the current (unordered) set Λ_k of values λ^s, and the sequence $\bar{\Lambda}_k$ of those values λ^s that have already been ordered.

The intersection of ξ with the first facet of K^* can be found by the procedure given in [3], but a considerably more efficient procedure is described below as Algorithm 2.

Algorithm 1

0. Find (by Algorithm 2) λ^1 such that $\bar{x} - a\lambda^1 = \xi \cap \mathrm{bd}(K^*)$, and the associated vector δ^1. Set $k = 1$, $\Lambda_1 = \{\lambda^1\}$, and go to 1.

1. If $\Lambda_k = \emptyset$, stop: the pth value λ^p in the sequence $\bar{\Lambda}_{k-1}$ defines the intersection of ξ with the pth (extended) facet of K^*, and all the (extended) facets met by ξ are listed.

Otherwise, choose

$$(13) \qquad \lambda^k = \min\{\lambda^s \mid \lambda^s \in \Lambda_k\}$$

(ties can be broken arbitrarily). If $\lambda^k = \lambda^{k-1}$ and $\delta^k = \delta^{k-1}$, remove λ^k from Λ_k and repeat step 1. Otherwise (i.e., if $\lambda^k > \lambda^{k-1}$, or $\lambda^k = \lambda^{k-1}$ but $\delta^k \neq \delta^{k-1}$), add λ^k to the tail end of the sequence $\bar{\Lambda}_{k-1}$, to obtain

$$(14) \qquad \bar{\Lambda}_k = \{\bar{\Lambda}_{k-1}, \lambda^k\}$$

(where $\bar{\Lambda}_0 = \emptyset$ by definition). Then define

$$(15) \qquad \lambda_i^k = \frac{\varphi_k - 2\bar{x}_i\delta_i^k}{\alpha_k - 2a_i\delta_i^k}, \qquad i \in N_k^* = \{i \in N \mid \delta_i^k = \delta_i^1\},$$

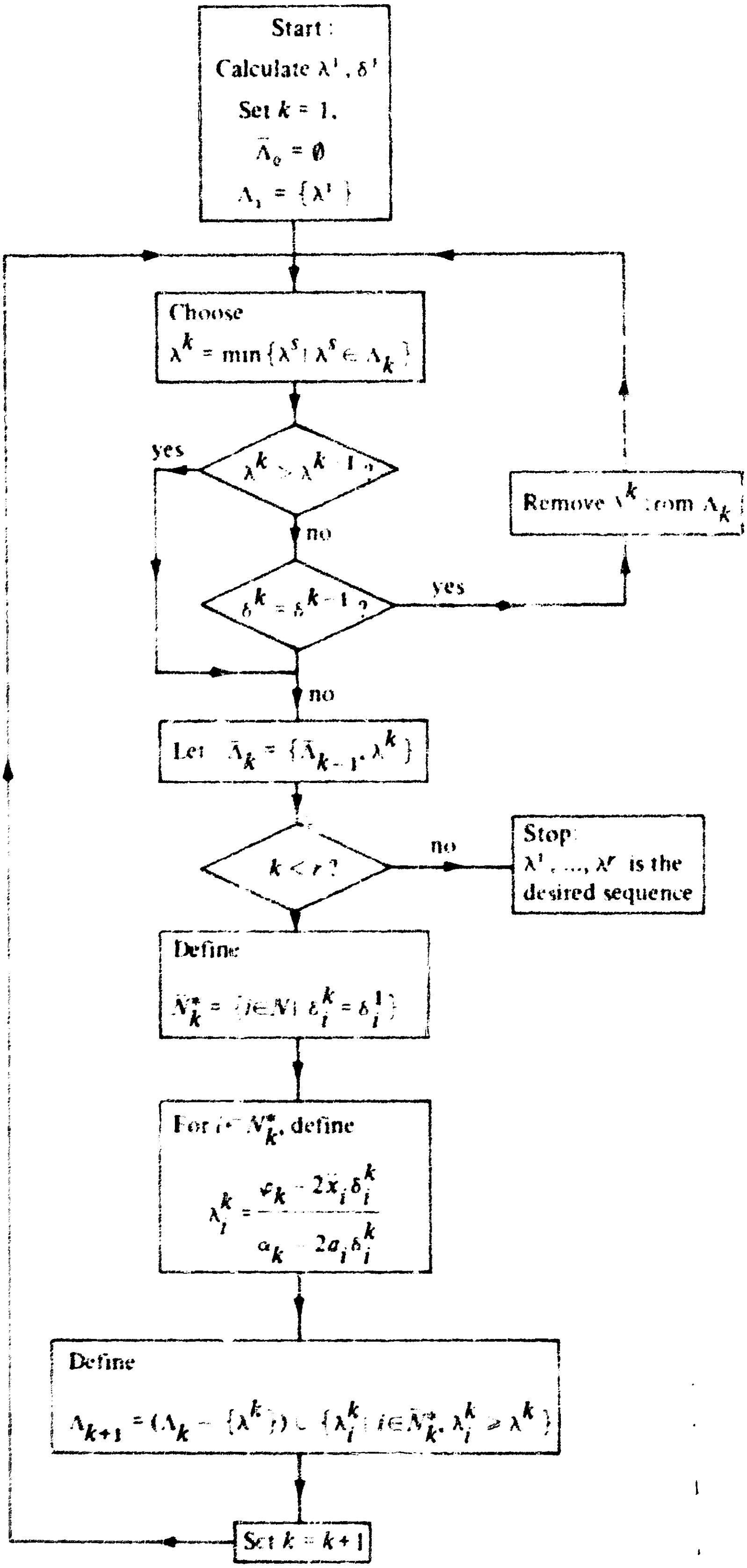

Fig. 1. Flowchart of Algorithm 1.

and set

(16) $$\Lambda_{k+1} = (\Lambda_k - \{\lambda^k\}) \cup \{\lambda_i^k \mid i \in \bar{N}_k^*, \lambda_i^k \geq \lambda^k\}.$$

Set $k = k+1$ and go to 1.

Theorem 2. *Algorithm 1 ranks the (extended) facets of K^* intersected by ξ in the order in which they are intersected.*

Proof. Let $\bar{\Lambda}_r = (\lambda^1, \ldots, \lambda^r)$ be the sequence obtained by the algorithm, i.e. assume the algorithm stops at the $(r+1)$st iteration. Each term λ^k of the sequence $\bar{\Lambda}_r$ defines the intersection of ξ with the (extended) facet f_k of K^* whose equation is $\delta^k x = \frac{1}{2}n$. Further, the sequence $\bar{\Lambda}_r$ is by construction nondecreasing, since each term λ^k is the smallest element of Λ_k, and all sets Λ_s such that $s > k$ contain either elements that are also contained in Λ_k, or elements λ^q such that $\lambda^q \geq \lambda^k$. This implies that, if f_h is intersected before f_k, i.e., if $\lambda^h < \lambda^k$, then λ^h precedes λ^k in the sequence $\bar{\Lambda}_r$. Further, whenever a λ^k, considered for ordering, is found to be equal to a λ^h already ordered, the possibility of duplicating a facet is excluded by checking that $\delta^k \neq \delta^h$, thus no facet can be listed twice.

It remains to be shown that f_r is indeed the rth (extended) facet intersected by ξ, i.e., no facets are missed. We show this by contradiction. Let f_s, defined by $\delta^s x = \frac{1}{2}n$, be an (extended) facet of K^* intersected by ξ before f_r but missed by the algorithm. Then, according to Theorem 1, there exists a sequence of $t+1$ (extended) facets $f_{j(0)}, f_{j(1)}, \ldots, f_{j(t)}$, with associated values $\lambda^{j(0)}, \lambda^{j(1)}, \ldots, \lambda^{j(t)}$, such that

(i) $\lambda^{j(h)} \geq \lambda^{j(h-1)}$, $h = 1, \ldots, t$,

(ii) $f_{j(h)}$ is adjacent to $f_{j(h-1)}$, $h = 1, \ldots, t$,

(iii) $f_{j(0)} = f_1$ (i.e., $f_{j(0)}$ is the first facet intersected by ξ), $f_{j(t)} = f_s$, and δ^s differs from δ^1 (defining f_1) by exactly t components.

We shall now show that if an arbitrary term $\lambda^{j(k)}$ of the sequence $\lambda^{j(0)}, \ldots, \lambda^{j(t-1)}$ is generated (i.e., is not missed) by the algorithm, then $\lambda^{j(k+1)}$ is also generated (not missed). Since the first term of the above sequence, $\lambda^{j(0)} = \lambda^1$, is never missed, it then follows that the last term, f_s, can also not be missed, contrary to the assumption. This proves that f_r is indeed the rth (extended) facet intersected by ξ.

Let $\lambda^{j(k)}$ be a term, generated by the algorithm, of the sequence

$\lambda^{j(0)}, \ldots, \lambda^{j(i-1)}$. Then from (i), $\lambda^{j(k+1)} > \lambda^{j(k)}$; from (ii), $\lambda^{j(k+1)}$ is of the form $\lambda_i^{j(k)}$ (defined by (15)) for some $i \in N$; whereas from (iii), $\lambda^{j(k+1)} = \lambda_i^{j(k)}$ for some $i \in \bar{N}^*_{j(k)}$. Hence

$$\lambda^{j(k+1)} \in \{\lambda_i^{j(k)} \mid i \in \bar{N}^*_{j(k)},\ \lambda_i^{j(k)} \geq \lambda^{j(k)}\},$$

which proves that $\lambda^{j(k+1)}$ is generated by the algorithm if $\lambda^{j(k)}$ is generated.

Fig. 1 illustrates Algorithm 1 for the case when one wants to find the intersection of ξ with the first r (extended) facets of K^* $(1 < r < 2^{n-1})$, intersected by ξ.

§3. Finding the facet intersected first

Next we describe an algorithm for finding the intersection of ξ with the boundary of K^*, i.e., with the first (extended) facet of K^*. This algorithm usually requires considerably fewer steps than the one given in [3]. (For another procedure that finds the intersection of ξ with the first facet and a few subsequent facets see [5].)

Algorithm 2

0. Start. Set $k = 1$ and

$$(17)\qquad \begin{aligned} N^+_{01} &= \{i \in N \mid a_i < 0; \ \text{ or } \ a_i = 0, \bar{x}_i > 0\}, \\ N^-_{01} &= \{i \in N \mid a_i > 0; \ \text{ or } \ a_i = 0, \bar{x}_i \leq 0\}. \end{aligned}$$

Go to 1.

1. Let $\delta^k = (\delta_i^k)$ be the n-vector defined by

$$(18)\qquad \delta_i^k = \begin{cases} 1 & \text{if } \ i \in N^+_{0k}, \\ -1 & \text{if } \ i \in N^-_{0k}. \end{cases}$$

Define

(19) $$\lambda^{0k} = \frac{\bar{x}\delta^k - \frac{1}{2}n}{a\delta^k}$$

and go to 2.

2. Let

(20) $$N_{0k}^* = \{i \in N \mid a_i\delta_i^k < 0,\ \bar{x}_i/a_i > \lambda^{0k}\}\ .$$

If $N_{0k}^* = \emptyset$, stop: $\lambda^{0k} = \lambda^0$. Otherwise, set

(21) $$\begin{aligned} N_{0,k+1}^+ &= (N_{0k}^+ \sim N_{0k}^*) \cup (N_{0k} \cap N_{0k}^*)\ ,\\ N_{0,k+1} &= (N_{0k} \sim N_{0k}^*) \cup (N_{0k}^+ \cap N_{0k}^*)\ . \end{aligned}$$

Set $k = k+1$, and go to 1.

Fig. 2 illustrates the algorithm

Theorem 3. *In r iterations, $1 \leq r < n$, Algorithm 2 finds $\lambda^{0r} = \lambda^0$ such that $\bar{x} - a\lambda^0 = \xi \cap \mathrm{bd}(K^*)$.*

Proof. We prove the theorem in 3 steps:

(i). First we show that $\lambda^{0k} > \lambda^{0,k+1}$, for all k such that $N_{0k}^* \neq \emptyset$. Denoting $\lambda^{0k} = \varphi_k/\alpha_k$, we note that $\alpha_1 < 0$. Assuming $\alpha_h < 0$ for $h = 1, \ldots, k$, we will show that $\alpha_{k+1} < 0$. From the definition of N_{0k}^* one has

$$\frac{\bar{x}_i\delta_i^k}{a_i\delta_i^k} > \frac{\varphi_k}{\alpha_k}\ , \qquad \forall i \in N_{0k}^*\ ,$$

or, multiplying through with $a_i\delta_i^k\alpha_k > 0$ (since $\alpha_k < 0$ and, from the definition of N_{0k}^*, $a_i\delta_i^k < 0$),

(22) $$\bar{x}_i\delta_i^k\alpha_k > a_i\delta_i^k\varphi_k\ , \qquad \forall i \in N_{0k}^*\ .$$

Adding up the inequalities of (22), multiplying the resulting inequality by -2, and then adding $\varphi_k\alpha_k$ to both sides yields

$$(\varphi_k - 2\sum_{i\in N_{0k}^*} \bar{x}_i\delta_i^k)\alpha_k < (\alpha_k - 2\sum_{i\in N_{0k}^*} a_i\delta_i^k)\varphi_k\ ,$$

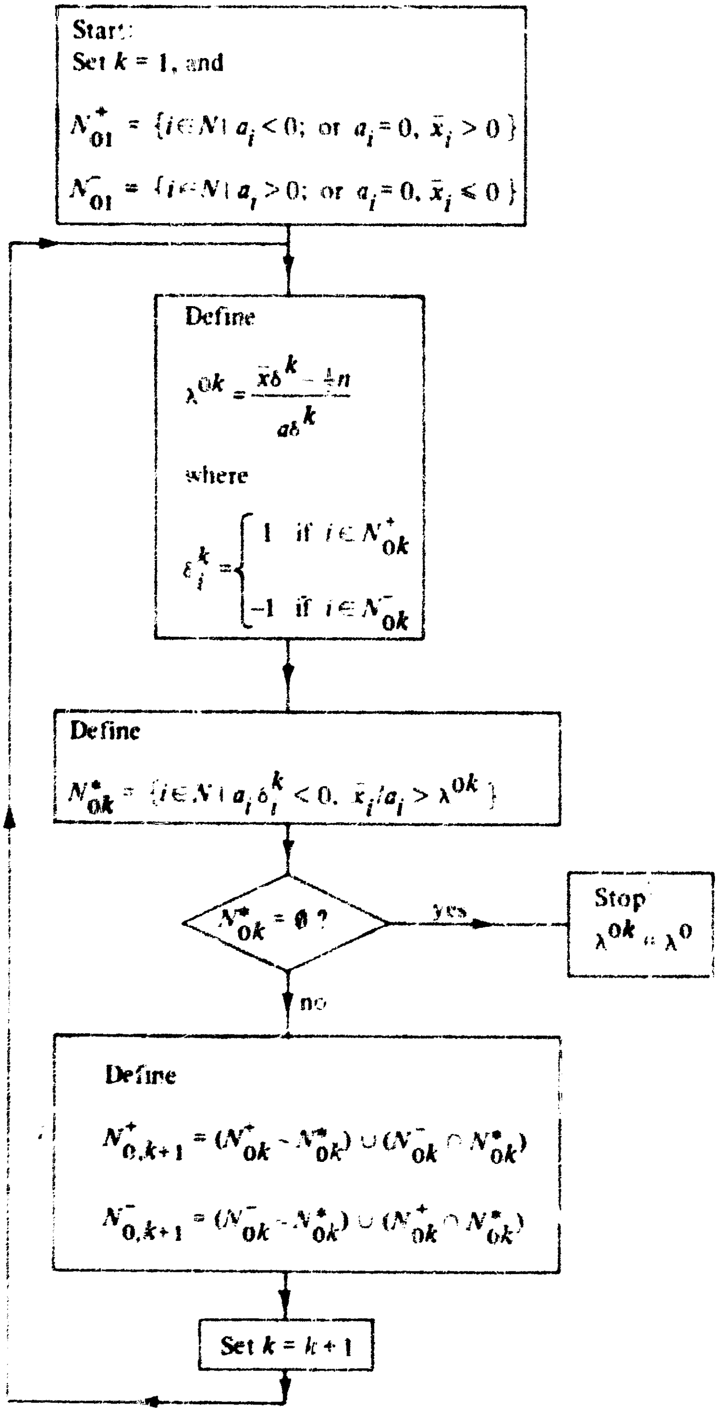

Fig. 2. Flow-chart of Algorithm 2.

or

$$\varphi_{k+1}\alpha_k < \alpha_{k+1}\varphi_k \tag{23}$$

(since, from (21), the vector δ^{k+1} defining $\lambda^{0,k+1} = \varphi_{k+1}/\alpha_{k+1}$ is obtained from δ^k by changing the sign of the components δ_i^k, $i \in N_{0k}^{*}$).

Since $\alpha_k < 0$, $\varphi_k < 0$, and $\varphi_{k+1} < 0$ (for the same reason that $\varphi_k < 0$), from (23) it follows that $\alpha_{k+1} < 0$. Dividing (23) by $\alpha_k \alpha_{k+1} > 0$ then yields

$$\lambda^{0k} = \frac{\varphi_k}{\alpha_k} > \frac{\varphi_{k+1}}{\alpha_{k+1}} = \lambda^{0,k+1} .$$

(ii). Next we show that $N^*_{0k} = N^2(\lambda^{0k}) = \{i \in N \mid 0 < \lambda_i^{0k} < \lambda^{0k}\}$, where, as before,

$$\lambda_i^{0k} = \frac{\varphi_k - 2\bar{x}_i \delta_i^k}{\alpha_k - 2a_i \delta_i^k} .$$

Since $\varphi_k - 2\bar{x}_i\delta_i^k < 0$, $\lambda_i^{0k} > 0$ if and only if $\alpha_k - 2a_i\delta_i^k < 0$. Further, multiplying the inequality $\lambda^{0k} > \lambda_i^{0k}$ by $\alpha_k(\alpha_k - 2a_i\delta_i^k) > 0$ and simplifying yields (22). Hence

$$N^2(\lambda^{0k}) = N_1^2(\lambda^{0k}) \cup N_2^2(\lambda^{0k}) ,$$

where

$$N_1^2(\lambda^{0k}) = \{i \in N \mid \bar{x}_i/a_i > \lambda^{0k}, a_i\delta_i^k < 0\} ,$$

$$N_2^2(\lambda^{0k}) = \{i \in N \mid \bar{x}_i/a_i < \lambda^{0k}, a_i\delta_i^k > 0\} .$$

But from (22), $N_1^2(\lambda^{0k}) = N^*_{0k}$, and we will show that $N_2^2(\lambda^{0k}) = \emptyset$. By the definition of N^*_{01} and N_{01}, $a^i\delta_i^1 < 0$, $\forall i \in N$; and under the rules of step 2, the sign of δ_i^1 is changed at some iteration h if and only if $\bar{x}_i/a_i > \lambda^{0h}$. But then, in view of (i), $\bar{x}_i/a_i > \lambda^{0k}$ for all $k > h$. Hence $\bar{x}_i/a_i < \lambda^{0k} \Rightarrow a_i\delta_i^k < 0$ and $N_2^2(\lambda^{0k}) = \emptyset$ for all k. Thus $N^2(\lambda^{0k}) = N_1^2(\lambda^{0k}) = N^*_{0k}$.

(iii). We are now ready to show that $\xi \cap \mathrm{bd}(K^*) = \bar{x} - a\lambda^{0r}$, and $r < n$, where r is defined as the smallest integer k for which $N^*_{0k} = \emptyset$. Since $N^*_{0k} = N^2(\lambda^{0k})$, r is the smallest integer k such that $N^2(\lambda^{0k}) = \emptyset$. But then, from Corollary 1.1, $\bar{x} - a\lambda^{0r} = \xi \cap \mathrm{bd}(K^*)$. Also, since (21) is such that no $i \in N$ can belong to more than one set N^*_{0k}, it follows that $r < n$.

Remark. Since thé recursion (21) changes the sign of *all* δ_i^k, $i \in N^2(\lambda^{0k})$, rather than of just one, r is usually much smaller than n (actually, in most cases it is 1 or 2).

A computer program written by Christopher Piper has made it possible to test Algorithm 2 on a number of randomly generated problems. Forty-seven cases were run with n between 20 and 27, with the following results:

Number of iterations required:	1	2	3	> 3
Number of cases:	14	24	6	3

§4. Numerical example

Find the intersection of $\xi = \{x \mid x = \bar{x} - a\lambda,\ \lambda \geq 0\}$ with the first 5 (extended) facets of K^*, where

$$\bar{x} = (-0.5, 0.4, -0.5, -0.3, 0, 0.4, 0.1, -0.5, -0.5),$$

$$a = (0, 0.8, 0, -0.7, 0.5, 0.3, -0.3, -1, 0).$$

I. First we use Algorithm 2 to find λ^1 such that $\bar{x} - a\lambda^1 = \xi \cap \mathrm{bd}(K^*)$:
0. $N_{01}^* = \{4,7,8\}$, $N_{01} = \{1,2,3,5,6,9\}$; $\delta^1 = (-1,-1,-1,1,-1,1,1,1,-1)$.
1. $\lambda^{01} = (\bar{x}\delta^1 - 4.5)/(a\delta^1) = (-4.5)/(-3.6)$; $N_{01}^* = \{6\}$.
2. $N_{02}^* = \{4,6,7,8\}$, $N_{02} = \{1,2,3,5,9\}$; $\delta^2 = (-1,-1,-1,1,-1,1,1,1,-1)$;
$\lambda^{02} = (-3.7)/(-3.0)$, $N_{02}^* = \emptyset$, stop: $\lambda^1 = \lambda^{02} = (-3.7)/(-3)$.

II. Next we use Algorithm 1 to find the values λ defining the intersection of ξ with the next 4 (extended) facets of K^*:

0. $\lambda^1 = (-3.7)/(-3)$, $\delta^1 = (-1,-1,-1,1,-1,1,1,1,-1)$;
$\bar{\Lambda}_0 = \emptyset$; $\Lambda_1 = \{\lambda^1\}$; $k = 1$.
1. $\lambda^1 = \min\{\lambda^s \mid \lambda^s \in \Lambda_1\} = \lambda^1$; $\bar{\Lambda}_1 = \{\lambda^1\}$; $\bar{N}_1^* = N = \{1, \dots, 9\}$;

$$\lambda_1^1 = \frac{-4.7}{-3},\ \lambda_2^1 = \frac{-2.9}{-1.4},\ \lambda_3^1 = \frac{-4.7}{-3},\ \lambda_4^1 = \frac{-3.1}{-1.6},\ \lambda_5^1 = \frac{-3.7}{-2},$$

$$\lambda_6^1 = \frac{-4.5}{-3.6},\ \lambda_7^1 = \frac{-3.72}{-2.4},\ \lambda_8^1 = \frac{-2.7}{-1},\ \lambda_9^1 = \frac{-4.7}{-3};$$

$\lambda_i^1 \geq \lambda^1$, $\forall i \in \bar{N}_i^*$;
$\Lambda_2 = \{\lambda_i^1 \mid i=1, \dots, 9\}$; $k = 2$.
2. $\lambda^2 = \min\{\lambda^s \mid \lambda^s \in \Lambda_2\} = \lambda_6^1 = (-4.5)/(-3.6)$; $\delta^2 = (-1,-1,-1,1,-1,-1,1,1,-1)$;
$\bar{\Lambda}_2 = \{\lambda^1, \lambda_6^1\}$; $\bar{N}_2^* = N - \{6\}$;

$$\lambda_1^2 = \frac{-5.5}{-3.6},\ \lambda_2^2 = \frac{-3.7}{-2},\ \lambda_3^2 = \frac{-5.5}{-3.6},\ \lambda_4^2 = \frac{-3.9}{-2.2},\ \lambda_5^2 = \frac{-4.5}{-2.6}.$$

$\lambda_7^2 = \frac{-4.52}{-3}$, $\lambda_8^2 = \frac{-3.5}{-1.6}$, $\lambda_9^2 = \frac{-5.5}{-3.6}$.

$\lambda_i^2 \geq \lambda^2$, $i \neq 6$;

$\Lambda_3 = \{\lambda_i^1 \mid i \neq 6\} \cup \{\lambda_i^2 \mid i \neq 6\}$; $k = 3$.

3. $\lambda^3 = \lambda_7^2 = (-4.52)/(-3)$; $\delta^3 = (-1,-1,-1,1,-1,-1,-1,1,-1)$;

$\Lambda_3 = \{\lambda^1, \lambda_6^1, \lambda_7^2\}$; $N_3^* = N - \{6,7\}$;

$\lambda_1^3 = \frac{-5.52}{-3}$, $\lambda_2^3 = \frac{-4.44}{-1.4}$, $\lambda_3^3 = \frac{-5.52}{-3}$, $\lambda_4^3 = \frac{-4.46}{-1.6}$, $\lambda_5^3 = \frac{-4.52}{-2}$,

$\lambda_8^3 = \frac{-3.52}{-1}$, $\lambda_9^3 = \frac{-5.52}{-3}$;

$\lambda_i^3 \geq \lambda^3$, $i \neq 6,7$;

$\lambda_4 = \{\lambda_i^1 \mid i \neq 6\} \cup \{\lambda_i^2 \mid i \neq 6,7\} \cup \{\lambda_i^3 \mid i \neq 6,7\}$; $k = 4$.

4. $\lambda^4 = \lambda_9^2 = (-5.5)/(-3.6)$; $\delta^4 = (-1,-1,-1,1,-1,-1,1,1,1)$;

$\Lambda_4 = \{\lambda^1, \lambda_6^1, \lambda_7^2, \lambda_9^2\}$; $N_4^* = N - \{6,9\}$;

$\lambda_1^4 = \frac{-6.5}{-3.6}$, $\lambda_2^4 = \frac{-4.7}{-2}$, $\lambda_3^4 = \frac{-6.5}{-3.6}$, $\lambda_4^4 = \frac{-4.9}{-2.2}$, $\lambda_5^4 = \frac{-5.5}{-2.6}$,

$\lambda_7^4 = \frac{-5.52}{-3}$, $\lambda_8^4 = \frac{-4.5}{-1.6}$

$\lambda_i^4 \geq \lambda^4$, $i \neq 6,9$;

$\Lambda_5 = \{\lambda_i^1 \mid i \neq 6\} \cup \{\lambda_i^2 \mid i \neq 6,7,9\} \cup \{\lambda_i^3 \mid i \neq 6,7\} \cup \{\lambda_i^4 \mid i \neq 6,9\}$; $k = 5$.

5. $\lambda^5 = \lambda_3^2 = (-5.5)/(-3.6)$; $\delta^5 = (-1,-1,1,1,-1,-1,1,1,-1)$;

$\Lambda_5 = \{\lambda^1, \lambda_6^1, \lambda_7^2, \lambda_9^2, \lambda_3^2\}$ defines the first 5 (extended) facets intersected by ξ_i.

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